Group Test I

Name: Solutions

1. Suppose that $A = UDV^T$ where U and V are $n \times n$ matrices with the property $U^T U = I$ and $V^T V = I$, and where D is a diagonal matrix with positive numbers on the diagonal.

   a. Prove that A is invertible..

   ~~b. Find an expression for A^{-1} (justify each step)~~.

(a) Given: (1) $A = UDV^T$

(2) U & V are $n \times n$ matrices

(3) $U^T U = I$

(4) $V^T V = I$

(5) D is invertible

Prove: (1) U is invertible (since $U^T U = I$)

$\rightarrow$ i.e. $U^{-1} = U^T$ and $(U^T)^{-1} = U$

(2) V is invertible (since $V^T V = I$)

$\rightarrow$ i.e. $V^{-1} = V^T$ and $(V^T)^{-1} = V$

(3) Let $Q = UD$, this implies that

Q is inertible since UD is invertible (theorem)

(4) Therefore $A = QV^T$ is invertible since

Q and V^T are invertible. Q.E.D

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2. Suppose that T is a linear transformation that maps $\Re^n$ onto $\Re^n$.

   a. Prove that T^{-1} exists and maps $\Re^n$ onto $\Re^n$.
   b. Prove that T^{-1} is one-to-one.

a) Given: (1) T is a linear transformation (i.e. $T(\vec{x}) = A\vec{x}$)

(2) T maps $\mathbb{R}^n$ onto $\mathbb{R}^n$

Proof: (1) T maps $\mathbb{R}^n$ to $\mathbb{R}^n$ iff the columns of A span $\mathbb{R}^n$ (Th 12, p89)

(2) A is $n \times n$ (i.e. square matrix) since $T: \mathbb{R}^n \to \mathbb{R}^n$

(2) Therefore A is invertible (Th 8h, p 129)

(3) Therefore T is invertible (Th 9, p131)

Q.E.D.

b) Given: (1) T is a linear transformation (i.e. $T(\vec{x}) = A\vec{x}$)

(2) T maps $\mathbb{R}^n \to \mathbb{R}^n$

(3) The columns of A span $\mathbb{R}^n$ (from above)

(4) A is $n \times n$ and invertible (from above)

Proof: (1) Therefore columns of A are linearly independent. (Th 8e, p. 129)

(2) Therefore T is one-to-one (Th 12b, p89)

Q.E.D.

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3. Suppose that A, X, and $A - AX$ are invertible, and suppose that $(A - AX)^{-1} = X^{-1}B$.

   a. Prove that B is invertible.

   ~~b. Solve for X (justify each step). – Hint~~

a.) Given: (1) $A, X, A-AX$ are invertible

(2) $(A-AX)^{-1} = X^{-1}B$

Proof (1) Therefore $X(A-AX)^{-1} = XX^{-1}B$ (left mult)

(2) Therefore $B = X(A-AX)^{-1}$. (identity property)

(3) Therefore B is invertible since X and $(A-Ax)^{-1}$ is invertible (Th 6 by 121)

Q.E.D.

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4. Suppose vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ span $\Re^4$ and that $T(\vec{v}_1) = T(\vec{v}_2) = T(\vec{v}_3) = T(\vec{v}_4) = 0$.

   a. Prove that T is the zero transformation, that is $T(\vec{x}) = 0$ for all $x \in \Re^4$.

Given: ① $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ span $\mathbb{R}^4$

② $T(\vec{v}_1) = T(\vec{v}_2) = T(\vec{v}_3) = T(\vec{v}_4) = \vec{0}$

Proof: ① Any vector $\vec{x} \in \mathbb{R}^4$ can be writen as a linear combination of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$, namely $\vec{x} = a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 + d\vec{v}_4$

(Definition, p 35)

② Therefore $T(\vec{x}) = T(a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 + d\vec{v}_4)$

(since $\vec{x} = a\vec{v}_1 + \dots$)

③ Therefore $T(\vec{x}) = aT(\vec{v}_1) + bT(\vec{v}_2) + cT(\vec{v}_3) + dT(\vec{v}_4)$

(property of linear transform)

④ $T(\vec{x}) = a(0) + b(0) + c(0) + d(0) = 0$

(from Given)

QED