Score:

Name:____________________________

Section (circle one): 1 2 3 4 5 6
Team (circle one): a b c d e f

SM261 –Test #1– Fall 2009

1. (10 pts) Let $A=\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}$ and $B=\begin{bmatrix}1&2\\3&4\end{bmatrix}$ and $\vec{x}=\begin{bmatrix}1\\2\end{bmatrix}$. Find the following if possible.

(**<u>Do not use your calculator – show all steps</u>**)

a. $A\vec{x}$

$$\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} \Rightarrow \text{can not do this!}$$

$$(2\times3)\quad(2\times1)$$

b. $B\vec{x}$

$$\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = 1\begin{bmatrix}1\\3\end{bmatrix}+2\begin{bmatrix}2\\4\end{bmatrix} = \begin{bmatrix}1+4\\3+8\end{bmatrix} = \begin{bmatrix}5\\11\end{bmatrix}$$

c. AB

$$\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\begin{bmatrix}1&2\\3&4\end{bmatrix} \longrightarrow \text{can not do this!!}$$

$$(2\times3)\quad(2\times2)$$

d. A^TB

$$\begin{bmatrix}1&4\\2&5\\3&6\end{bmatrix}\begin{bmatrix}1&2\\3&4\end{bmatrix} = \begin{bmatrix}1+12&2+16\\2+15&4+20\\3+18&6+24\end{bmatrix}$$

$$(3\times2)\quad(2\times2)$$

$$= \begin{bmatrix}13&18\\17&24\\21&30\end{bmatrix}$$

Prob	Pts	Score
1	10	
2	20	
3	20	
4	10	
5	10	
6	10	
GT	20	
$\sum$	100	

2. (20 pts) Consider the linear system below where k is a real number:

$$\begin{aligned} x + 2y + z &= 3 \\ 2x + 6z + 4w &= 2 \\ 5x + 2y + 13z + 8w &= k \end{aligned}$$

a. For which values of k are does the system have:
   (i) no solutions (ii) infinite solutions

$$\left[\begin{array}{ccccc} 1 & 2 & 1 & 0 & 3 \\ 2 & 0 & 6 & 4 & 2 \\ 5 & 2 & 13 & 8 & k \end{array}\right] \begin{array}{l} \\ R2-2R1 \\ R3-5R1 \end{array} \Rightarrow \left[\begin{array}{ccccc} 1 & 2 & 1 & 0 & 3 \\ 0 & -4 & 4 & 4 & -4 \\ 0 & -8 & 8 & 8 & k-15 \end{array}\right] R2 \div -4$$

$$\Rightarrow \left[\begin{array}{ccccc} 1 & 2 & 1 & 0 & 3 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & -8 & 8 & 8 & k-15 \end{array}\right] R3+8R2 \Rightarrow \left[\begin{array}{ccccc} 1 & 2 & 1 & 0 & 3 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & k-7 \end{array}\right]$$

(i) no solution $k \neq 7$

(ii) infinite solutions $k = 7$

b. Assume $k = 7$, find all solutions of the systems and write them in vector form (**<u>You may use your calculator</u>**).

pivot cols — free variables z, w

$$\left[\begin{array}{ccccc} 1 & 2 & 1 & 0 & 3 \\ 2 & 0 & 6 & 4 & 2 \\ 5 & 2 & 13 & 8 & 7 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccccc} 1 & 0 & 3 & 2 & 1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x + 3z + 2w = 1 \Rightarrow x = 1 - 3z - 2w$

$y - z - w = 1 \Rightarrow y = 1 + z + w$

$z = z$

$w = w$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

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3. (20 pts) Alka-Seltzer contains sodium bicarbonate ($NaHCO_3$) and citric acid ($H_3C_6H_5O_7$). When a tablet is dissolved in water it produces sodium citrate, water and carbon dioxide. Using techniques from matrix algebra to balance the following chemical formula. (**<u>You may use your calculator on the problem</u>**)

a. <u>3</u> $NaHCO_3$ + <u>1</u> $H_3C_6H_5O_7$ → $Na_3C_6H_5O_7$ + <u>3</u> H_20 + <u>3</u> CO2

$$\begin{bmatrix} Na \\ H \\ C \\ O \end{bmatrix} \quad a\begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix} + b\begin{bmatrix} 0 \\ 8 \\ 6 \\ 7 \end{bmatrix} = c\begin{bmatrix} 3 \\ 5 \\ 6 \\ 7 \end{bmatrix} + d\begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} + e\begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -3 & 0 & 0 \\ 1 & 8 & -5 & -2 & 0 \\ 1 & 6 & -6 & 0 & -1 \\ 3 & 7 & -7 & -1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

e free variable

$$\Rightarrow \begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 0 \\ 1 & 8 & -5 & -2 & 0 & 0 \\ 1 & 6 & -6 & 0 & -1 & 0 \\ 3 & 7 & -7 & -1 & -2 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1/3 & 0 \\ 0 & 0 & 1 & 0 & -1/3 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

$a = e$
$b = 1/3 e$
$c = 1/3 e$
$d = e$

let $e = 3 \Rightarrow$

$a = 3$
$b = 1$
$c = 1$
$d = 3$
$e = 3$

$$3NaHCO_3 + H_3C_6H_5O_7 \rightarrow Na_3C_6H_5O_7 + 3H_2O + 3CO_2$$

4. (10 pts) Suppose A, B and C are 2×2 matrices. $ABC = I_2$

a. Find A^{-1} if $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$.

$$\frac{1}{6-5}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

b. Find $(AB)^{-1}$ if $B^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$

$$(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

c. Find C If $ABC = I_2$.

$$\Rightarrow C = (AB)^{-1} I_2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

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5. (10 points) Use row reductions methods to find the inverses of the following matrices. **(<u>Do not use your calculator – show all steps.</u>)**

a. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] R3-R1 \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] R2-R3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

b. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] R3-R1 = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

No inverse, Columns of matrix not linearly independent

6. (10 points) Suppose vectors $\{\vec{v}_1, \vec{v}_2\}$ span $\Re^2$ and that $T(\vec{v}_1) = T(\vec{v}_2) = 0$. ***Prove that** T **is the zero transformation, that is** $T(\vec{x}) = 0$ **for all** $x \in \Re^2$:*

Given: ① $\{\vec{v}_1, \vec{v}_2\}$ span $\mathbb{R}^2$

② $T(\vec{v}_1)=0$, $T(\vec{v}_2)=0$

③ $x \in \mathbb{R}^2$

Proof ① Since $\{\vec{v}_1, \vec{v}_2\}$ span $\mathbb{R}^2$ and $\vec{x} \in \mathbb{R}^2$

then $\vec{x} = a\vec{v}_1 + b\vec{v}_2$ (definition, p 35)

② $T(\vec{x}) = T(a\vec{v}_1 + b\vec{v}_2)$ (since $x = a\vec{v}_1 + b\vec{v}_2$)

③ $T(a\vec{v}_1 + b\vec{v}_2) = aT(\vec{v}_1) + bT(\vec{v}_2)$

(property of linear transform)

④ $T(\vec{x}) = a(0) + b(0) = 0$ for all $x \in \mathbb{R}^2$

Q.E.D.