Score:

Name: Solutions

Section (circle one): 1 2 3 4 5 6
Team (circle one): a b c d e f

SM261 –Test #3– Fall 2009

Closed Book. One Theorem Sheet. Calculators Allowed. Turn in Theorem Sheet with Test.

1. Answer the following:
   a. (5 pts) Is $\lambda = 4$ an eigenvalues of $A = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$? Why or why not?

$$\det(A-\lambda I) = \left|\begin{bmatrix} 10-4 & -9 \\ 4 & -2-4 \end{bmatrix}\right| = \left|\begin{bmatrix} 6 & -9 \\ 4 & -6 \end{bmatrix}\right| = -36+36=0$$

$\therefore$ $\lambda=4$ is an eigenvalue

   b. (5 pts) Is $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ an eigenvector of $A = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$? Why or why not?

$$\begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10-9 \\ 4-2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \leftarrow \text{this is not a multiple of } \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\therefore$ $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is <u>not</u> a eigenvector

   c. (5 pts) Find the eigenvalues for $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$.

$\lambda = 1, 4, 6$ (since A is upper triangular)

   a. (5 pts) If λ is an eigenvalue for A, then show that $1/\lambda$ is an eigenvalue for A^{-1}

$$A\vec{v} = \lambda\vec{v} \Rightarrow \underbrace{A^{-1}A}_{I}\vec{v} = A^{-1}\lambda\vec{v}$$

$$\Rightarrow \vec{v} = \lambda A^{-1}\vec{v} \Rightarrow \boxed{A^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}}$$

Prob	Pts	Score
1	20	
2	20	
3	10	
4	20	
5	10	
GT	20	
$\sum$	100	

2. Answer the following:

a. (10 pts) Find the eigenvalues and eigenvectors for $A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}$.

$$\det\begin{bmatrix} 7-\lambda & 4 \\ -3 & -1-\lambda \end{bmatrix} = -7+7\lambda+\lambda+\lambda^2+12=0$$

$$\Rightarrow \lambda^2-6\lambda+5=0 \Rightarrow (\lambda-5)(\lambda-1)=0 \Rightarrow \boxed{\lambda=5,\ \lambda=1}$$

$$(\lambda=1)\quad \begin{bmatrix} 6 & 4 \\ -3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0 \Rightarrow 6x_1+4x_2=0 \Rightarrow x_2=-\frac{3}{2}x_1 \Rightarrow x_1=2,\ x_2=-3 \Rightarrow \boxed{V_1=\begin{bmatrix} 2 \\ -3 \end{bmatrix}}$$

$$\lambda=5\quad \begin{bmatrix} 2 & 4 \\ -3 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0 \Rightarrow 2x_1+4x_2=0 \Rightarrow x_2=-\frac{1}{2}x_1 \Rightarrow x_1=2,\ x_2=-1 \quad \boxed{V_2=\begin{bmatrix} 2 \\ -1 \end{bmatrix}}$$

b. (5 pts) Diagonalize $A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}$.

$$A = PDP^{-1} = \begin{bmatrix} 2 & 2 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}\frac{1}{4}\begin{bmatrix} -1 & -2 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} -1/4 & -1/2 \\ 3/4 & 1/2 \end{bmatrix}$$

c. (5 pts) If $AP = PD$, derive a general expression for A^n.

$$\Rightarrow A = PDP^{-1}$$

$$\Rightarrow A^n = (PDP^{-1})(PDP^{-1})\ldots\ldots P^{-1})(PDP^{-1}) \quad \text{"n-times"}$$

$$= PD\underbrace{(P^{-1}P)}_{I}D\underbrace{(P^{-1}P)}_{I}\ldots\ldots\underbrace{(P^{-1}P)}_{I}DP^{-1} \quad \text{"(n-times)}$$

$$\Rightarrow \boxed{A^n = PD^nP^{-1}}$$

Name:______________________________

3. (10 pts) Given that $\vec{x}_{k+1} = A\vec{x}_k$ where $A = \begin{bmatrix} 1.7 & -.3 \\ -1.2 & .8 \end{bmatrix}$ with eigenvalues $\lambda_1 = 2$ and $\lambda_2 = .5$. How does the system behave as k gets large? What is the direction of greatest repulsion or attraction?

$$\vec{x}_n = c_1 \lambda_1^n \vec{v}_1 + c_2 \lambda_2^n \vec{v}_2$$

$$\lambda_1 = 2 \Rightarrow \begin{bmatrix} -.3 & -.3 \\ -1.2 & -1.2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow v_1 = \begin{bmatrix} +1 \\ -1 \end{bmatrix}$$

$$\lambda_2 = .5 \Rightarrow \begin{bmatrix} 1.2 & -.3 \\ -1.2 & .3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow 1.2x_1 - .3x_2 = 0 \Rightarrow x_2 = 4x_1 \qquad v_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\therefore \vec{x}_n = c_1 (2)^n \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 (.5)^n \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

as n get large, this 'goes away'

This term goes to infinite

$$\therefore \lim_{k\to\infty} x_k = \begin{bmatrix} \infty \\ -\infty \end{bmatrix}$$

direction of increase is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

4. W is a plane in $\mathbb{R}^3$ is spanned by the vectors $\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\-1\end{bmatrix}\right\}$. $\vec{y} = \begin{bmatrix}3\\2\\1\end{bmatrix}$ is a vector in $\mathbb{R}^3$.

vectors are orthogonal

a. (10 pts) Find $proj_W\,\vec{y}$.

$$\text{proj}_W \vec{y} = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

$$c_1 = \frac{[1\ 2\ 1]\cdot[3\ 2\ 1]}{[1\ 2\ 1]\cdot[1\ 2\ 1]} = \frac{8}{6} = \frac{4}{3}$$

$$c_2 = \frac{[-1\ 1\ -1]\cdot[3\ 2\ 1]}{[-1\ 1\ -1]\cdot[-1\ 1\ -1]} = \frac{-2}{3}$$

$$\text{proj}_W \vec{y} = \frac{4}{3}\begin{bmatrix}1\\2\\1\end{bmatrix} - \frac{2}{3}\begin{bmatrix}-1\\1\\-1\end{bmatrix} = \begin{bmatrix}2\\2\\2\end{bmatrix}$$

b. (10 pts) Find $proj_{W^\perp}\,\vec{y}$.

$$\text{proj}_{W^\perp} \vec{y} = \vec{y} - \text{proj}_W \vec{y} = \begin{bmatrix}3\\2\\1\end{bmatrix} - \begin{bmatrix}2\\2\\2\end{bmatrix} = \begin{bmatrix}1\\0\\-1\end{bmatrix}$$

Name:______________________________

5. Given the data on the right.

x	y
0	1
1	2
3	3

a. (5 pts) Find the line, $y = mx + b$, of best fit.

$$\underset{A}{\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} m \\ b \end{bmatrix}} = \underset{\vec{b}}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} \Rightarrow \begin{bmatrix} 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 10 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 11 \\ 6 \end{bmatrix} \Rightarrow \text{rref} \begin{bmatrix} 10 & 4 & 11 \\ 4 & 3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 9/14 \\ 0 & 1 & 8/7 \end{bmatrix} \Rightarrow \boxed{y = \frac{9}{14}x + \frac{8}{7}}$$

b. (5 pts) Find the parabola, $y = ax^2 + bx + c$, that fits this points.

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 9 & 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \text{rref} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 9 & 3 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -1/6 \\ 0 & 1 & 0 & 7/6 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\therefore \boxed{y = -\frac{1}{6}x^2 + \frac{7}{6}x + 1}$$